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Integrability of a $t - J$ model with impurities

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Abstract. A $t - J$ model for correlated electrons with impurities is proposed. The impurities are introduced in such a way that integrability of the model in one dimension is not violated. The algebraic Bethe ansatz solution of the model is also given and it is shown that the Bethe states are highest weight states with respect to the supersymmetry algebra $gl(2|1)$.

1. Introduction

The quantum inverse scattering method (QISM) has lead to many new results in the study of integrable and exactly solvable systems. Amongst these is the fact that the $t - J$ model for correlated electrons is integrable in one dimension at the supersymmetric point $J = 2t$ with the supersymmetry algebra given by the Lie superalgebra $gl(2|1)$. This was made apparent in the works [1, 2] where it was shown that the Hamiltonian could be derived from a solution of the Yang–Baxter equation. Also, solutions of the model were found by means of the algebraic Bethe ansatz.

One attractive aspect of the quantum inverse scattering method is that one is allowed to incorporate impurities into the system without violating integrability. In this context, several versions of the Heisenberg chain with impurities have been investigated [3–5]. For the specific case of the $t - J$ model this idea was first adopted by Bares [6], whereby the impurities were introduced into the model by way of inhomogeneities in the transfer matrix of the system. Another possibility was explored by Bedürftig *et al* [7] with impurities given by changing the representation of the $gl(2|1)$ generators at some lattice sites from the fundamental three-dimensional representation to the one-parameter family of typical four-dimensional representations which were introduced in [8] to derive the supersymmetric U model.

Here we wish to propose a third method for introducing integrable impurities into the $t-J$ model. This is achieved by replacing some lattice sites with the *dual* space of the fundamental three-dimensional representation. A significant point here is that only recently have new Bethe ansatz methods been proposed in order to solve such a system because of the lack of a suitable (unique) reference state. Rather, one is forced to work with a subspace of reference states. This approach has been developed in the works of Abad and Ríos [9, 10] and has already been adopted in [11] to find a Bethe ansatz solution of the supersymmetric U model starting from a ferromagnetic space of states.

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The Hamiltonian of this $t-J$ model with impurities reads

$$H=\sum_{i=1}^{L}h_{i,i+1}+\sum_{i\in I}\frac{2}{\lambda_i-2}h_{i,i+1}Q_i-\frac{2}{\lambda_i}Q_ih_{i,i+1} \tag{1}$$

where

$$\begin{aligned}h_{i,i+1}=&-\sum_{\sigma}(c^{\dagger}_{i,\sigma}c_{i+1,\sigma}+c^{\dagger}_{i+1,\sigma}c_{i,\sigma})(1-n_{i,-\sigma})(1-n_{i+1,-\sigma})\\ &+2\big(\boldsymbol{S}_i\cdot\boldsymbol{S}_{i+1}-\tfrac{1}{4}n_in_{i+1}\big)+n_i+n_{i+1}-1\\ Q_i=&\sum_{\sigma}\sigma(c^{\dagger}_{i,\sigma}c_{\sigma}-c^{\dagger}_{\sigma}c_{i,\sigma})(1-n_{i,-\sigma})n_{-\sigma}\\ &+S_i^+S^-+S_i^-S^++n_{i+}n_-+n_{i-}n_+-n+1\end{aligned}$$

and periodic boundary conditions are imposed. Above $c^{(\dagger)}_{i\pm}$ are spin-up or spin-down annihilation (creation) operators, the $\boldsymbol{S}_i$ spin matrices, the n_i occupation numbers of electrons at lattice site i. The λ_i are arbitrary complex parameters and I is simply an index set with elements in the range $1,2,\ldots,L$. We make the assumption that if $i\in I$ then $i\pm 1\notin I$, since otherwise extra terms are needed in the Hamiltonian for integrability. The operators without site labels in the expression for Q_i act on the impurity space coupled to the site i. Note, however, that the interactions involving the impurity sites are three site interactions involving the sites i and $i+1$ as well as the impurity. The local space of states for an impurity site has the basis

$$|\uparrow\rangle,\quad |\downarrow\rangle,\quad |\uparrow\downarrow\rangle$$

in contrast to the local spaces for the other sites which have bases

$$|\uparrow\rangle,\quad |\downarrow\rangle,\quad |0\rangle$$

as is the case for a pure $t-J$ model. The reason for this choice is so that the Hamiltonian conserves magnetization and particle number. Finally, we mention that the first term in equation (1) is the Hamiltonian for the pure $t-J$ model. We can recover this model from equation (1) by taking the limit $\lambda_i\to\infty$ for each $i\in I$.

In this paper we derive the Hamiltonian equation (1) by means of the QISM which guarantees integrability. We will also find solutions to the model using the algebraic Bethe ansatz. Finally, we also show that the Bethe states which are obtained by this procedure are, in fact, highest weight states with respect to the underlying supersymmetry algebra $gl(2|1)$.

2. Derivation of the Hamiltonian

Recall that the Lie superalgebra $gl(m|n)$ has generators $\{E^i_j\}_{i,j=1}^{m+n}$ satisfying the commutation relations

$$[E^i_j,E^k_l]=\delta^k_jE^i_l-(-1)^{([i]+[j])([k]+[l])}\delta^i_lE^k_j \tag{2}$$

where the $\mathbb{Z}_2$-grading on the indices is determined by

$$\begin{aligned}[i]&=0 &&\text{for}\quad 1\leqslant i\leqslant m\\ [i]&=1 &&\text{for}\quad m<i\leqslant m+n.\end{aligned}$$

This induces a $\mathbb{Z}_2$-grading on the $gl(m|n)$ generators through

$$\big[E^i_j\big]=[i]+[j]\ (\mathrm{mod}\ 2).$$

The vector module V has basis $\{v^i\}_{i=1}^{m+n}$ with action defined by

$$E^i_j v^k = \delta^k_j v^i. \tag{3}$$

Associated with this space there is a solution $R(u) \in \mathrm{End}(V \otimes V)$ of the Yang–Baxter equation

$$R_{12}(u-v)\, R_{13}(u)\, R_{23}(v) = R_{23}(v)\, R_{13}(u)\, R_{12}(u-v) \tag{4}$$

on the space $V \otimes V \otimes V$, which is given by

$$R(u) = I \otimes I - \frac{2}{u} \sum_{i,j} e^i_j \otimes e^j_i (-1)^{[j]}. \tag{5}$$

Above, the matrices e^i_j have elements given by $(e^i_j)_{kl} = \delta_{ik}\delta_{jl}$. We remark that equation (4) is acting on a supersymmetric space so the multiplication of tensor products is governed by the relation

$$(a \otimes b)(c \otimes d) = (-1)^{[b][c]} ac \otimes bd \tag{6}$$

for homogeneous operators b, c.

The solution given by equation (5) allows us to construct a universal L-operator which reads

$$L(u) = I \otimes I - \frac{2}{u} \sum_{i,j} e^i_j \otimes E^j_i (-1)^{[j]}. \tag{7}$$

This operator gives us a solution of the Yang–Baxter equation of the form

$$R_{12}(u-v)\, L_{13}(u)\, L_{23}(v) = L_{23}(v)\, L_{13}(u)\, R_{12}(u-v)$$

on the space $V \otimes V \otimes gl(m|n)$, which follows from the commutation relations equation (2). The dual representation to equation (3) acts on the module V^* with basis $\{v_i\}_{i=1}^{m+n}$ and the action is given by

$$E^i_j v_k = -(-1)^{[i]+[i][j]} \delta^i_k v_j. \tag{8}$$

By taking this representation in the expression equation (7) we obtain the following R-matrix:

$$R^*(u) = I \otimes I + \frac{2}{u} \sum_{i,j} e^i_j \otimes e^i_j (-1)^{[i][j]} \tag{9}$$

giving the solution

$$R_{12}(u-v)\, R^*_{13}(u)\, R^*_{23}(v) = R^*_{23}(v)\, R^*_{13}(u)\, R_{12}(u-v) \tag{10}$$

on $V \otimes V \otimes V^*$.

We wish to construct an impurity model with generic quantum spaces represented by V and the impurity spaces by V^* for the case when the supersymmetry algebra is $gl(2|1)$. To this end take some index set $I = \{p_1, p_2, \ldots, p_l\},\ 1 \leqslant p_i \leqslant L$ and define

$$W = \bigotimes_{i=1}^{L} W_i$$

where

$$\begin{aligned} W_i &= V && \text{if} \quad i \notin I \\ W_i &= V \otimes V^* && \text{if} \quad i \in I. \end{aligned} \tag{11}$$

In other words, for each $i \in I$ we are coupling an impurity into the lattice which will be situated between the sites i and $i+1$.

Next we define the monodromy matrix

$$T(u, \{\lambda\}) = \overline{R}_{01}(u)\, \overline{R}_{02}(u) \ldots \overline{R}_{0L}(u)$$

where we have

$$\overline{R}_{0i}(u) = R_{0i}(u) \qquad \text{for} \quad i \notin I$$
$$\overline{R}_{0i}(u) = R_{0i'}(u)\, R^*_{0i''}(u - \lambda_i) \qquad \text{for} \quad i \in I.$$

Above, the indices i' and i'' refer to the two spaces in W_i (cf equation (11)) and the λ_i are arbitrary complex parameters. A consequence of equations (4) and (10) is that the monodromy matrix satisfies the intertwining relation

$$R_{12}(u - v)\, T_{13}(u)\, T_{23}(v) = T_{23}(v)\, T_{13}(u)\, R_{12}(u - v) \tag{12}$$

acting on the space $V \otimes V \otimes W$. The transfer matrix is defined by

$$\tau(u) = \mathrm{tr}_0\, \sigma_0 T(u) \tag{13}$$

where the matrix σ has entries

$$\sigma^i_j = (-1)^{[i][j]}\delta^i_j$$

from which the Hamiltonian is obtained through

$$H = -2\frac{\mathrm{d}}{\mathrm{d}u} \ln(u^L \tau(u))\bigg|_{u=0}. \tag{14}$$

In this derivation we have used the property

$$Q_i P_{i,i+1} Q_i = Q_i$$

which follows from the fact that Q projects onto a one-dimensional space spanned by the vector

$$v^1 \otimes v_1 + v^2 \otimes v_2 + v^3 \otimes v_3.$$

Above, P is the $\mathbb{Z}_2$-graded permutation operator defined by

$$P(x \otimes y) = (-1)^{[x][y]} y \otimes x$$

for any homogeneous vectors x, y and extends to inhomogeneous vectors linearly. This simplifies the calculations and is one of the reasons why this model is much simpler than other impurity chains. From equation (12) we conclude by the usual argument that the transfer matrix provides a set of abelian symmetries for the model and hence the Hamiltonian is integrable. In the next section we will solve the model by the algebraic Bethe ansatz approach. The explicit form of the Hamiltonian equation (1) is given by making the following identification between the basis elements of V, V^* and the electronic states:

$$v^1 = |\uparrow\rangle \qquad v_1 = |\downarrow\rangle$$
$$v^2 = |\downarrow\rangle \qquad v_2 = |\uparrow\rangle$$
$$v^3 = |0\rangle \qquad v_3 = |\uparrow\downarrow\rangle.$$

3. Algebraic Bethe ansatz solution

By a suitable redefinition of the matrix elements, the solutions (5) and (9) may be written in terms of operators which satisfy the Yang–Baxter equations (4) and (10) without $\mathbb{Z}_2$-grading (see e.g. [12]). These operators read

$$R(u) = \sum_{i,j} e^i_i \otimes e^j_j (-1)^{[i][j]} - \frac{2}{u} e^i_j \otimes e^j_i$$
$$R^*(u) = \sum_{i,j} e^i_i \otimes e^j_j (-1)^{[i][j]} + \frac{2}{u} e^i_j \otimes e^i_j (-1)^{[i]+[j]}$$

and hereafter we will use these forms. In the following we will also need the R-matrices

$$r(u) = \sum_{i,j=2}^{3} (-1)^{[i][j]} e_i^i \otimes e_j^j - \frac{2}{u} e_j^i \otimes e_i^j$$

$$r^*(u) = \sum_{i,j=2}^{3} (-1)^{[i][j]} e_i^i \otimes e_j^j + \frac{2}{u} e_j^i \otimes e_j^i (-1)^{[i]+[j]}$$

which belong to a $gl(1|1)$-invariant (six-vertex) system. From these matrices we define the monodromy matrices

$$t(v,\{u\}) = r_{01}(v-u_1)\, r_{02}(v-u_2) \ldots r_{0N}(v-u_N)$$
$$t^*(v,\{\lambda\}) = r^*_{01}(v-\lambda_1)\, r^*_{02}(v-\lambda_2) \ldots r^*_{0l}(v-\lambda_l).$$

First we construct the Yangian algebra which has elements $\{Y_j^i(u)\}_{i,j=1}^{m+n}$. Relations amongst these elements are governed by the constraint

$$R_{12}(u-v)\, Y_{13}(u)\, Y_{23}(v) = Y_{23}(v)\, Y_{13}(u)\, R_{12}(u-v) \tag{15}$$

where

$$Y(u) = \sum_{i,j} e_j^i \otimes Y_i^j(u).$$

By comparison with equation (12) we see that the monodromy matrix provides a representation of this algebra acting on the module W by the mapping

$$\pi(Y_j^i(u))_l^k = (-1)^{([i][l]+[j][l]+[i][k])} T_{jl}^{ik}(u). \tag{16}$$

Moreover, the transfer matrix is expressible in terms of this representation by

$$\tau(u) = \sum_{i=1}^{3} (-1)^{[i]+[i][k]} \pi(Y_i^i(u))_l^k.$$

The phase factors present above are required since the Yangian algebra is defined with a non-graded R-matrix. In the following we will omit the symbol π for ease of notation.

For a given $\{\alpha\} = (\alpha_1, \alpha_2, \ldots, \alpha_l)$, $\alpha_i = 2, 3$ we define the vector $v^{\{\alpha\}} \in W$ by

$$v^{\{\alpha\}} = \bigotimes_{i=1}^{L} w^i$$

where

$$w^i = v^1 \qquad \text{for} \quad i \notin I$$
$$w^i = v^1 \otimes v_{\alpha_j} \qquad \text{for} \quad i = p_j \in I.$$

Now set $X = \text{span}\,\{v^{\{\alpha\}}\}$. It is important to observe that the space X is closed under the action of the elements $Y_j^i(u)$, $i, j = 2, 3$ which generate a sub-Yangian. We may, in fact, write

$$Y_j^i(u)\, v^{\{\alpha\}} = t^{*i\{\alpha\}}_{j\{\alpha'\}}(u,\{\lambda\})\, v^{\{\alpha'\}}$$

which follows from the fact that the $Y_j^i(u)$ $(i, j = 2, 3)$ act trivially on the vector v^1 in the sense

$$Y_2^2(u)\, v^1 = Y_3^3(u)\, v^1 = v^1$$
$$Y_3^2(u)\, v^1 = Y_2^3(u)\, v^1 = 0.$$

Setting

$$S^{\{\beta\}}(\{u\}) = Y_1^{\beta_1}(u_1)\, Y_1^{\beta_2}(u_2) \ldots Y_1^{\beta_N}(u_N) \qquad \beta_i = 2, 3$$

we look for a set of eigenstates of the transfer matrix of the form

$$\Phi^j = \sum_{\{\beta,\alpha\}} S^{\{\beta\}}(\{u\})\, v^{\{\alpha\}} F^j_{\{\beta,\alpha\}} \tag{17}$$

where the $F^j_{\{\beta,\alpha\}}$ are undetermined coefficients. We appeal to the algebraic equations given by equation (15) to determine the constraints on the variables u_i needed to force equation (17) to be an eigenstate. Although many relations occur as a result of equation (15) only the following are required:

$$Y_1^1(v)\, Y_1^\beta(u) = a(u-v)\, Y_1^\beta(u)\, Y_1^1(v) - b(u-v)\, Y_1^\beta(v)\, Y_1^1(u) \tag{18}$$

$$Y_\gamma^{\gamma'}(v)\, Y_1^\alpha(u) = Y_1^{\alpha'}(u)\, Y_\gamma^{\gamma''}(v)\, r^{\gamma'\alpha}_{\gamma''\alpha'}(v-u) - b(v-u)\, Y_1^{\gamma'}(v)\, Y_\gamma^\alpha(u) \tag{19}$$

$$a(v-u)\, Y_1^\alpha(v)\, Y_1^\beta(u) = Y_1^{\beta'}(u)\, Y_1^{\alpha'}(v)\, r^{\beta\alpha}_{\beta'\alpha'}(v-u) \tag{20}$$

with $a(u) = 1 - 2/u$ and $b(u) = -2/u$. All of the indices in equations (18)–(20) assume only the values 2 and 3. Using equation (18) two types of term arise when Y_1^1 is commuted through Y_1^α. In the first type Y_1^1 and Y_1^α preserve their arguments and in the second type their arguments are exchanged. The first type of terms are called *wanted terms* because they will give a vector proportional to Φ^j, and the second type are *unwanted terms* (u.t.). We find that

$$Y_1^1(v)\,\Phi^j = a(v)^L \prod_{i=1}^N a(u_i - v)\,\Phi^j + \text{u.t.} \tag{21}$$

Similarly, for $i = 2, 3$ we have from equation (19) (no sum on i)

$$\begin{aligned} Y_i^i(v)\,\Phi^j &= S^{\{\beta'\}}(\{u\})\, Y_k^i(v)\, t^{k\{\beta\}}_{i\{\beta'\}}(v,\{u\})\, v^{\{\alpha\}} F^j_{\{\beta,\alpha\}} + \text{u.t.} \\ &= S^{\{\beta'\}}(\{u\})\, t^{k\{\beta\}}_{i\{\beta'\}}(v,\{u\})\, t^{*i\{\alpha\}}_{k\{\alpha'\}}(v,\{\lambda\})\, v^{\{\alpha'\}} F^j_{\{\beta,\alpha\}} + \text{u.t.} \\ &= S^{\{\beta'\}}(\{u\})\, \bar{t}^{i\{\beta,\alpha\}}_{i\{\beta',\alpha'\}}(v,\{u,\lambda\})\, v^{\{\alpha'\}} F^j_{\{\beta,\alpha\}} + \text{u.t.} \end{aligned}$$

where

$$\bar{t}^{i\{\beta,\alpha\}}_{i\{\beta',\alpha'\}}(v,\{u,\lambda\}) = t^{k\{\beta\}}_{i\{\beta'\}}(v,\{u\})\, t^{*i\{\alpha\}}_{k\{\alpha'\}}(v,\{\lambda\}).$$

The contribution to the eigenvalues of the transfer matrix is

$$\begin{aligned} Y_2^2(v)\,\Phi^j + (-1)^{1+[j]}\, Y_3^3(v)\,\Phi^j &= \sum_{i=2}^3 (-1)^{[i]+[i][j]}\, \bar{t}^{i\{\beta,\alpha\}}_{i\{\beta',\alpha'\}} \\ &\quad \times (v,\{u,\lambda\}) S^{\{\beta'\}}(\{u\})\, v^{\{\alpha'\}} F^j_{\{\beta,\alpha\}} + \text{u.t.} \end{aligned} \tag{22}$$

At this point we need to perform a second-level, or *nested* Bethe ansatz procedure to diagonalize the matrix

$$\tau_1(v)^{\{\beta,\alpha\}}_{\{\beta',\alpha'\}} = \sum_{i=2}^3 (-1)^{[i]+[i][\{\beta,\alpha\}]} \bar{t}^{i\{\beta,\alpha\}}_{i\{\beta',\alpha'\}}(v,\{u,\lambda\})$$

where we have used the fact that $F^j_{\{\beta,\alpha\}} = 0$ unless $[j] = [\{\beta,\alpha\}]$. The above matrix is simply the transfer matrix for a $gl(1|1)$-invariant system acting in the tensor product representation of N copies of the vector representation with inhomogeneities $\{u\}$ and l copies of the dual representation with inhomogeneities $\{\lambda\}$.

To diagonalize this matrix we construct the Yangian generated by

$$y(u) = \sum_{i,j=2}^3 e^i_j \otimes y^j_i(u)$$

subject to the constraint

$$r_{12}(u-v)\,y_{13}(u)\,y_{23}(v) = y_{23}(v)\,y_{13}(u)\,r_{12}(u-v). \tag{23}$$

From the above set of relations we will need the following:

$$y_2^2(v)\,y_2^3(u) = a(u-v)\,y_2^3(u)\,y_2^2(v) - b(u-v)\,y_2^3(v)\,y_2^2(u) \tag{24}$$

$$y_3^3(v)\,y_2^3(u) = -a(u-v)\,y_2^3(u)\,y_3^3(v) - b(v-u)\,y_2^3(v)\,y_3^3(u) \tag{25}$$

$$y_2^3(v)\,y_2^3(u) = \frac{-a(u-v)}{a(v-u)}y_2^3(u)\,y_2^3(v). \tag{26}$$

Proceeding similarly to before, we look for eigenstates of the form

$$\phi = y_2^3(\gamma_1)\,y_2^3(\gamma_2)\ldots y_2^3(\gamma_M)\,w$$

with the vector w given by

$$w = S^{\{2\}}(\{u\})\,v^{\{3\}}.$$

Using (24) and (25) it follows that

$$\tau_1(v)\,\phi = \Lambda_1(v)\,\phi + \text{u.t.}$$

with

$$\Lambda_1(v) = \prod_{i=1}^{N} a(v-u_i)\prod_{k=1}^{M} a(\gamma_k - v) - \prod_{j=1}^{l} a(v-\lambda_j)\prod_{k=1}^{M} a(\gamma_k - v).$$

The unwanted terms cancel provided the parameters γ_k satisfy the Bethe ansatz equations (BAE)

$$\prod_{i=1}^{N} a(\gamma_k - u_i) = \prod_{j=1}^{l} a(\gamma_k - \lambda_j) \qquad k = 1, 2, \ldots, M. \tag{27}$$

Combining this result with equation (21) we obtain for the eigenvalues of the transfer matrix equation (13)

$$\Lambda(v) = a(v)^L \prod_{i=1}^{N} a(u_i - v) + \Lambda_1(v). \tag{28}$$

Cancellation of the unwanted terms in (21) and (22) leads to a second set of BAE which are

$$a(u_h)^L \prod_{i=1}^{N} \frac{a(u_i - u_h)}{a(u_h - u_i)} = -\prod_{k=1}^{M} a(\gamma_k - u_h) \qquad h = 1, 2, \ldots, N. \tag{29}$$

We will not give the details proving the cancellation of the unwanted terms but remark that the calculation is analogous to that given in [2] for the pure $t-J$ chain.

Making a change of variable $u \to \mathrm{i}u + 1$, $\gamma \to \mathrm{i}\gamma + 2$, $\lambda \to \mathrm{i}\lambda + 1$ the BAE read

$$-\left(\frac{u_h + \mathrm{i}}{u_h - \mathrm{i}}\right)^L = \prod_{i=1}^{N} \frac{u_h - u_i + 2\mathrm{i}}{u_h - u_i - 2\mathrm{i}} \prod_{k=1}^{M} \frac{u_h - \gamma_k - \mathrm{i}}{u_h - \gamma_k + \mathrm{i}} \qquad h = 1, \ldots, N \tag{30}$$

$$\prod_{i=1}^{N} \frac{\gamma_k - u_i + \mathrm{i}}{\gamma_k - u_i - \mathrm{i}} = \prod_{j=1}^{l} \frac{\gamma_k - \lambda_j + \mathrm{i}}{\gamma_k - \lambda_j - \mathrm{i}} \qquad k = 1, \ldots, M. \tag{31}$$

In the absence of impurities (limit $l \to 0$) we recover the form of the BAE first derived by Sutherland [13] and later by Sarkar [14] for the usual $t-J$ model. Adopting the string conjecture, or more specifically assuming that the solutions u_i are real or appear as complex conjugate pairs and the λ_j are real, we find string solutions

$$u^n_{\alpha\beta} = u^n_\alpha + \mathrm{i}(n + 1 - 2\beta) \qquad \alpha = 1, 2, \ldots, N_n \qquad \beta = 1, 2, \ldots, n \qquad n = 1, 2, \ldots$$

and the γ_k are real. The number of n-strings N_n satisfy the relation

$$N = \sum_n n N_n.$$

As was shown in the papers [1, 2] two other forms of the Bethe ansatz exist which are obtained by choosing a different grading for the indices of the $gl(2|1)$ generators. Recall that the above calculations were performed with the choice

$$[1] = [2] = 0 \qquad [3] = 1.$$

Choosing

$$[1] = 1 \qquad [2] = [3] = 0$$

yields the eigenvalue expression

$$\Lambda(v) = -a(-v)^L \prod_{i=1}^N a(v - u_i) + \prod_{k=1}^M a(v - \gamma_k) \prod_{j=1}^l a(\lambda_j - v) + \prod_{i=1}^N a(v - u_i) \prod_{k=1}^M a(\gamma_k - v)$$

subject to the BAE

$$a(-u_i)^L = \prod_{k=1}^M a(\gamma_k - u_i) \qquad i = 1, 2, \ldots, N$$

$$\prod_{k=1}^M \frac{a(\gamma_h - \gamma_k)}{a(\gamma_k - \gamma_h)} = -\prod_{i=1}^N a(\gamma_h - u_i) \prod_{j=1}^l \frac{1}{a(\lambda_j - \gamma_h)} \qquad h = 1, 2, \ldots, M.$$

In the limit $l \to 0$ we recover Lai's form of the BAE [15] (see also [16]). Alternatively, choosing

$$[1] = [3] = 0 \qquad [2] = 1$$

yields the eigenvalue expression

$$\Lambda(v) = a(v)^L \prod_{i=1}^N a(u_i - v) + \prod_{k=1}^M a(v - \gamma_k) \prod_{j=1}^l a(\lambda_j - v) - \prod_{i=1}^N a(u_i - v) \prod_{k=1}^M a(v - \gamma_k)$$

with the BAE

$$a(u_i)^L = \prod_{k=1}^M a(u_i - \gamma_k) \qquad i = 1, 2, \ldots, N,$$

$$\prod_{i=1}^N a(u_i - \gamma_k) = \prod_{j=1}^l a(\lambda_j - \gamma_k) \qquad k = 1, 2, \ldots, M.$$

Finally, from the definition of the Hamiltonian equation (14) we see that the energies are given by

$$E = -2 \frac{\mathrm{d}}{\mathrm{d}v} \ln \left(v^L \Lambda(v) \right) \Big|_{v=0}.$$

Using the eigenvalue expression equation (28) we obtain

$$E = L - 4 \sum_{i=1}^N \frac{1}{1 + u_i^2}$$

where the u_i are solutions to equations (30) and (31).

4. Highest weight property

Next we wish to show that the eigenstates constructed in the previous section are, in fact, highest weight states with respect to the underlying supersymmetry algebra $gl(2|1)$. The highest weight property of the Bethe states has been proved for many models, such as the Heisenberg chain [17] and its generalized version [18], the Kondo model [19], the usual $t-J$ model [2], the Hubbard chain [20–22] and its $gl(2|2)$ extension [23]. However, as far as we are aware it has never been shown before in the case where a subspace of reference states has been used in the Bethe ansatz procedure.

Let us begin by considering

$$E_3^2\Phi^j = \sum_{\{\beta,\alpha\}} E_3^2\, S^{\{\beta\}}(\{u\})\, v^{\{\alpha\}} F^j_{\{\beta,\alpha\}}.$$

By means of the nesting procedure we know that the coefficients $F^j_{\{\beta,\alpha\}}$ are such that we have the following identification of states:

$$S^{\{\beta\}}(\{u\})\, v^{\{\alpha\}} F^j_{\{\beta,\alpha\}} = y_2^3(\gamma_1)\, y_2^3(\gamma_2) \ldots y_2^3(\gamma_M)\, w$$

for a suitable solution of the BAE. By comparing equations (7), (23) and (16) it is possible to determine algebraic relations between the elements of the Yangian algebra and the supersymmetry algebra. For our purposes we need the following:

$$[E_3^2, y_2^3(u)]^\alpha_\beta = -y_2^2(u)^\alpha_\beta + y_3^3(u)^\alpha_\beta\, (-1)^{[\alpha]}. \tag{32}$$

Noting that $E_3^2 w = 0$ it is evident that we may write

$$E_3^2\, y_2^3(\gamma_1) \ldots y_2^3(\gamma_M)\, w = \sum_{h=1}^{M} x_h X_h$$

with

$$X_h = y_2^3(\gamma_1) \ldots y_2^3(\gamma_{h-1})\, y_2^3(\gamma_{h+1}) \ldots y_2^3(\gamma_M)\, w$$

and the x_h some yet to be determined coefficients. To find x_h we write

$$y_2^3(\gamma_1) \ldots y_2^3(\gamma_M)\, w = \prod_{j=1}^{h-1} \frac{-a(\gamma_h-\gamma_j)}{a(\gamma_j-\gamma_h)} y_2^3(\gamma_h) X_h$$

where we have used equation (26). Now by using the relations (24), (25) and (32) and looking only for those terms which give a vector proportional to X_h we find that

$$x_h = \prod_{j=1}^{h-1} \frac{-a(\gamma_h-\gamma_j)}{a(\gamma_j-\gamma_h)} \left(\prod_{j=1}^{l} a(\gamma_h-\lambda_j) \prod_{k\neq h}^{M} a(\gamma_k-\gamma_h) - \prod_{i=1}^{N} a(\gamma_h-u_i) \prod_{k\neq h}^{M} a(\gamma_k-\gamma_h) \right)$$

which vanishes because of equation (27). Thus we see that

$$E_3^2\Phi^j = 0.$$

Next we consider the action of E_2^1 on Φ^j. Using equations (7), (15) and (16) we find the commutation relation

$$[E_2^1, Y_1^\alpha(u)] = \delta_2^\alpha\, Y_1^1(u) - Y_2^\alpha(u). \tag{33}$$

As before, since $E_2^1 v^{\{\alpha\}} = 0$ we can write the general expression

$$E_2^1\Phi^j = \sum_{h,\beta} z_{h,\beta} Z_{h,\beta}$$

where

$$Z_{h,\beta} = S^{\{\beta_h^-\}}(\{u_h^-\})\}) \, S^{\{\beta_h^+\}}(\{u_h^+\})\}) \, v^{\{\alpha\}} F^j_{\{\beta,\alpha\}}$$

and for any vector $\{w\}$ we have

$$\{w_h^-\} = (w_1, w_2, \ldots, w_{h-1}) \qquad \{w_h^+\} = (w_{h+1}, \ldots, w_N).$$

To calculate $z_{h,\beta}$ we begin by writing

$$\begin{aligned}\Phi^j &= S^{\{\beta_h^-\}}(\{u_h^-\}) \, Y_1^{\beta_h}(u_h) \, S^{\{\beta_h^+\}}(\{u_h^+\}) \, v^{\{\alpha\}} F^j_{\{\beta,\alpha\}} \\ &= \prod_{i=1}^{h-1} a(u_i - u_h)^{-1} t^{\beta_h \{\beta_h^-\}}_{\gamma\{\gamma_h^-\}}(-u_h, \{-u_h^-\}) \, Y_1^{\gamma}(u_h) \, S^{\{\gamma_h^-\}}(\{u_h^-\}) \, S^{\{\beta_h^+\}}(\{u_h^+\}) \, v^{\{\alpha\}} F^j_{\{\beta,\alpha\}}\end{aligned}$$

where we have used the relation equation (20). Now applying equation (33) and using the relations (18) and (19) to determine the terms which give a vector proportional to $Z_{h,\beta}$ we find that

$$z_{h,\beta} = \delta_2^{\beta_h} \left(a(u_h)^L \prod_{i \neq h}^{N} a(u_i - u_h) - \prod_{i \neq h}^{N} a(u_h - u_i) \prod_{k=1}^{M} a(\gamma_k - u_h) \right)$$

which vanishes as a result of equation (29). We then conclude that

$$E_2^1 \Phi^j = 0$$

which completes the proof that the Bethe states are $gl(2|1)$ highest weight states. We observe that this property can also be proved for the other two choices of gradings in a similar way.

5. Conclusions

In this paper we have introduced a new integrable version of the $t - J$ model with impurities. The model was solved through an algebraic Bethe ansatz method and three different forms of the BAE were derived. A proof of the highest weight property of the Bethe vectors with respect to the $gl(2|1)$ superalgebra was also presented. We believe that this is the first instance of the highest weight property being shown for a Bethe ansatz approach where there is no unique reference state.

After completing this work we became aware of the preprint [24] where a similar model has been studied for an alternating chain. It is worth noting that the explicit Hamiltonian that we have presented is significantly simpler than that of [24] because we make the restriction that impurities are not coupled to consecutive sites in the chain. We also received the preprint [25] where the braid-monoid algebra was used to analyse such types of impurity models.

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